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On the double Refraction of Light in a crystallized Medium, according to the Principles of Fresnel. By JAMES MAC CULLAGH, ESQ. *Communicated by the* REV. DR. SADLEIR, S. F. T. C. D., *Vice President.*

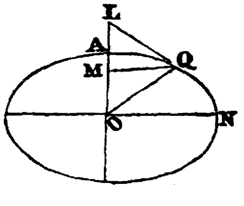
Read June 21st, 1830.

THE mathematical difficulties under which the beautiful and interesting theory of Fresnel has hitherto laboured, are well known, and have been regarded as almost insuperable. He tells us, in his Memoir, (see the Memoirs of the Royal Academy of Sciences of Paris, tom. vii. p. 136), that the calculations, by which he assured himself of the truth of his construction for finding the surface of the wave, were so tedious and embarrassing, that he was obliged to omit them altogether. A direct demonstration has since been supplied by M. Ampere, (Annales de Chimie et de Physique, tom. xxxix. p. 113) ; but his solution is excessively complicated and difficult.

Judging from the simplicity and elegance of the results, that there must be some simple method of arriving at them, I have been led to consider the subject with the attention which it merits, and have succeeded in discovering a method by which the whole may be explained with that simplicity which is characteristic of every theory that is founded in nature.

In the following paper I shall give a brief view of this method, sufficient to enable those who are acquainted with the mechanical principles laid down in the original memoir of Fresnel, to trace, at a glance, the connexion between the several parts of his theory. For this purpose it will be convenient to premise the following geometrical Lemmas. *Lemma* 1. If a, b, c, be the Semiaxes of an Ellipsoid, and α, β, γ, the angles which they make with a perpendicular from the centre on a tangent plane, the square of the perpendicular will be equal to

$$a^2 \cos.^2 \alpha + b^2 \cos.^2 \beta + c^2 \cos.^2 \gamma.$$



Let a plane through the point of contact Q and one of the semiaxes OA, intersect the ellipsoid in the ellipse AON, and the tangent plane in the tangent QL, and draw QM perpendicular to OA; then OA is a semiaxis of the ellipse AON, and therefore is a mean proportional between OM and OL; whence OL $= \frac{a^2}{x}$, denoting OM by x. But if p denote the length of the perpendicular from the centre O on the tangent plane at Q, the cosine of the angle which it makes with OA will be equal to $\frac{p}{\text{OL}}$, and therefore

$$\cos. \alpha = \frac{px}{a^2}, \text{ and } a \cos. \alpha = \frac{px}{a}.$$

Similarly

$$b \cos. \beta = \frac{py}{b}, \text{ and } c \cos. \gamma = \frac{pz}{c}.$$

Hence

$$a^2 \cos.^2 \alpha + b^2 \cos.^2 \beta + c^2 \cos.^2 \gamma = p^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = p^2.$$

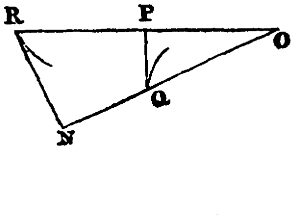
Cor. Since x, y, z, are as the cosines of the angles which OQ makes with the semiaxes, it appears from the demonstration that the cosines of the angles which the perpendicular to a tangent plane makes with the semiaxes are, with respect to each other, directly as the cosines of the angles which the semidiameter through the point of contact makes with the semiaxes, and inversely as the squares of the semiaxes themselves.

2. If the semiaxes a, b, c, and a', b', c', of two concentric ellipsoids, coincide in direction and be reciprocally proportional, so that

$aa' = bb' = cc' = k^2$; and if a semidiameter OR of the one be cut perpendicularly in P by a plane which touches the other, then will OR be inversely as OP, so that OP × OR will be always equal to k^2.

Let OR be a semidiameter of the ellipsoid whose semiaxes are a', b', c'; and let α, β, γ, be the angles which it makes with them; then if x, y, z, be the coordinates of R, we have

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1,$$



and therefore

$$\frac{1}{OR^2}=\frac{\cos.^2\alpha}{a'^2}+\frac{\cos.^2\beta}{b'^2}+\frac{\cos.^2\gamma}{c'^2}=$$

$$\frac{1}{k^4}(a^2\cos.^2\alpha+b^2\cos.^2\beta+c^2\cos.^2\gamma).$$

But by the preceding lemma, since OP is perpendicular to the tangent plane at Q, we have

$$OP^2=a^2\cos.^2\alpha+b^2\cos.^2\beta+c^2\cos.^2\gamma.$$

Hence

$$\frac{1}{OR^2}=\frac{OP^2}{k^4},$$

and therefore

$$OP\times OR=k^2.$$

3. If through the point of contact Q, the straight line OQ be drawn to meet in N the tangent plane at R, it will meet it at right angles.

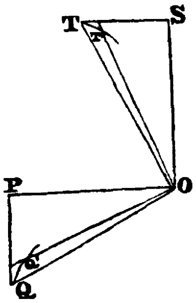
For the cosines of the angles made by OP with the semiaxes are directly as the cosines of the angles made by OQ with them, and inversely as the squares of a, b, c, (lem. 1. cor.); and the cosines of the angles made with the semiaxes by a perpendicular to the tangent plane at R, are directly as the cosines of the angles made with them by OP, and inversely as the squares of a', b', c', or as the same cosines and the squares of a, b, c, directly; that is, simply, as the cosines of the angles made by OQ with the semiaxes. Hence OQ coincides with the perpendicular to the tangent plane at R.

4. If a perpendicular at O to the plane POQ meet the surface of the ellipsoid *abc* in q, then will OQ and Oq be the semiaxes of the section QOq made by a plane passing through them.

For the tangent plane at Q and the plane QOq are perpendicular to POQ, and therefore the intersection of the two former, which is a tangent to the ellipse QOq at Q, is perpendicular to OQ; whence OQ is one semiaxis, and Oq the other.

If the perpendicular Oq meet the other ellipsoid in *r*, then OR and O*r* will be the semiaxes of the section ROr made by a plane passing through them; for (by lem. 3.) the straight line OQN is perpendicular to the tangent plane at R.

5. In a straight line at right angles to any diametral section QOq of the ellipsoid *abc*, let OT and OV be taken equal to OQ and Oq the semiaxes of the section, and imagine the double surface which is the locus of all the points T and V; then if OS be perpendicular to the plane which touches this surface in T, and OP to that which touches the ellipsoid in Q; the lines OP and OS will be equal and perpendicular to each other, and the four, OP, OQ, OS, OT, will lie in the same plane which will be at right angles to Oq.

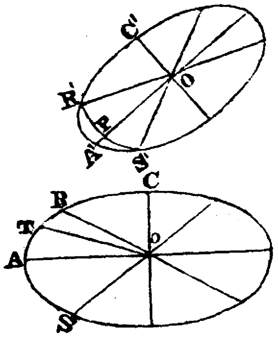


By the preceding lemma it is evident that Oq is perpendicular to the plane POQ, and since OT is perpendicular to the plane QOq, it follows that OP, OQ, OT, are in the same plane at right angles to Oq. In the surface which is the locus of T, and in the plane TOq, let a point T, be taken indefinitely near to T; then the plane of the section at right angles to OT, will pass through OQ, and will have one of its semiaxes (that to which OT, is equal) indefinitely near to OQ, and therefore differing from OQ by an indefinitely small quantity of the second order, adopting, for brevity, the language of infinitesimals. To see this, it is only necessary to recollect, that by the property of maxima and minima, the semiaxis differs from any semidiameter indefinitely near to it, such as OQ, by an indefinitely small quantity of the second order. Hence, since OT is equal to OQ, and OT, to the above-mentioned semiaxis, it follows that OT and OT, differ by an indefinitely small quantity of the second order, and that therefore the angle OTT, is ultimately a right angle: consequently the tangent to the curve in which the plane TOq intersects the locus of T is perpendicular to the plane PQOT. But the tangent plane at T passes through this tangent, and therefore the perpendicular OS to the tangent plane must lie in the plane PQOT.

Again, let a point T′ indefinitely near to T, and in the plane PQOT, be taken in the surface which is the locus of T, and let the plane of the section which is perpendicular to OT′ intersect the plane PQOT in the straight line OQ′ which meets the ellipsoid in Q′. Then that semiaxis of the section to which OT′ is equal, will be indefinitely near to OQ′ and will therefore differ from it by an indefinitely small quantity of the second order. Hence, since OT is equal to OQ, the angle OTT′ will be ultimately equal to OQQ′; and therefore, TS and QP being tangents, the angles OTS and OQP are equal. But OT = OQ, and the angles P and S are right, therefore OS = OP, and the angle SOT = POQ; whence SOP = TOQ = a right angle.

Similarly, if one perpendicular be let fall from O on a plane touching the locus in V, and another on the plane touching the ellipsoid in q, it may be proved that the two perpendiculars are equal and at right angles to each other, and that with the lines OV and Oq, they lie in a plane at right angles to OQ

6. An ellipsoid being cut by any plane through its centre, the difference between the squares of the reciprocals of the semiaxes of the section is proportional to the rectangle under the sines of the angles which the plane of the section makes with the planes of the two circular sections of the ellipsoid.—(See Fresnel's Memoir, p. 150.)



Let the plane which cuts the ellipsoid intersect its circular sections in the lines OR′, OS′, and let the principal section AOC of the ellipsoid cut the circular sections in OR and OS; then OR, OS, OR′, OS′, will be all equal to the mean semiaxis OB, and hence the semiaxes OA′, OC′, of the section A′OC′ will bisect the acute and obtuse angles made by OR′ and OS′. Let a plane through OB and OA′ intersect the principal plane AOC in the line OT. Then by the nature of the ellipse, we have, in the ellipse A′OC′,

$$\frac{1}{OR'^2} - \frac{1}{OA'^2} = \left(\frac{1}{OC'^2} - \frac{1}{OA'^2}\right) \text{sin.}^2\, A'OR';$$

and in the ellipse BOT,

$$\frac{1}{OB^2} - \frac{1}{OA'^2} = \left(\frac{1}{OB^2} - \frac{1}{OT^2}\right) \text{sin.}^2\, BOA'.$$

Hence, observing that OB and OR′ are equal, we have

$$\frac{1}{OC'^2} - \frac{1}{OA'^2} = \left(\frac{1}{OB^2} - \frac{1}{OT^2}\right) \frac{\text{sin.}^2\, BOA'}{\text{sin.}^2\, A'OR'}.$$

But the ellipse AOC gives

$$\frac{1}{OR^2} - \frac{1}{OT^2} = \left(\frac{1}{OC^2} - \frac{1}{OA^2}\right)(\text{sin.}^2\, AOR - \text{sin.}^2\, AOT) = \left(\frac{1}{OC^2} - \frac{1}{OA^2}\right) \times$$
$$\text{sin.}\, ROT\, \text{sin.}\, SOT.$$

Therefore

$$\frac{1}{OC'^2} - \frac{1}{OA'^2} = \left(\frac{1}{OC^2} - \frac{1}{OA^2}\right)\frac{\sin.^2 BOA'}{\sin.^2 A'OR'} \sin. ROT \sin. SOT.$$

because OB and OR are equal.

Imagine a sphere described from the centre O with a radius equal to OB, and passing through R′ and S′, and cutting A′O in P. The sides BP and PR′ of the spherical triangle BPR′ subtend the angles BOA′ and A′OR′ at the centre; and its angle PBR′ is equal to the angle ROT; whence the sines of the sides being porportional to the sines of the opposite spherical angles, it follows that

$$\frac{\sin. BOA'}{\sin. A'OR'} \sin. ROT$$

is equal to the sine of the spherical angle BR′P, which is the angle made by the section A′OC′ with the plane of the circular section BOR. Similarly, by means of the spherical triangle BPS′, it may be shewn that

$$\frac{\sin. BOA'}{\sin. A'OS'} \sin. SOT$$

is equal to the sine of the angle made by the plane A′OC′ with the plane of the circular section BOS. Therefore, since the angles A′OR′ and A′OS′ are equal, it follows that

$$\frac{1}{OC'^2} - \frac{1}{OA'} \text{ is equal to } \frac{1}{OC^2} - \frac{1}{OA^2}$$

multiplied by the product of the sines of the angles which the plane A′OC′ makes with the planes of the two circular sections.

I shall now demonstrate a geometrical construction for finding the magnitude and direction of the elastic force arising from a displacement in any direction—a construction, which, with the help of the preceding lemmas, will lead us immediately to all the conclusions established by Fresnel.

Let O be the position of a point in the medium when quiescent, and let three rectangular axes passing through it, and fixed in space, be taken for the axes of coordinates. For a small displacement in the direction of x, let the elastic forces, excited in the directions of x, y, z, be a, b, c; for an equal displacement in the direction of y let the forces be a', b', c'; and for the same in the direction of z let them be a'', b'', c''. Then if a point receive an equal displacement in a direction OI making with OX, OY, OR, the angles α, β, γ, the forces in the direction of x, y, z, (denoting then by X, Y, Z, respectively,) will be

$$X = a \cos.\alpha + a' \cos.\beta + a'' \cos.\gamma,$$
$$Y = b \cos.\alpha + b' \cos.\beta + b'' \cos.\gamma,$$
$$Z = c \cos.\alpha + c' \cos.\beta + c'' \cos.\gamma,$$

as follows from considering (see Fresnel's Memoir, p. 82.) that the force arising from a displacement in any direction is the resultant of the forces arising from the three displacements in the directions of x, y, z, which are the statical components of that displacement. But since (p. 90,) the force in the direction of one of the axes arising from a displacement in the direction of another, is equal to the force in the direction of the latter arising from an equal displacement in the direction of the former, it follows that $a' = b$, $a'' = c$, $b'' = c'$; and hence

$$X = a \cos.\alpha + b \cos.\beta + c \cos \gamma,$$
$$Y = b \cos.\alpha + b' \cos.\beta + c' \cos.\gamma,$$
$$Z = c \cos.\alpha + c' \cos.\beta + c'' \cos.\gamma.$$

Let $ax^2 + b'y^2 + c''z^2 + 2c'yz + 2cxz + 2bxy = 1$ be the equation of a surface of the second order. Let OI intersect it in I, the coordinates of I being x', y', z'; then the equation of the tangent plane at I will be, by the known formulæ

$$(ax' + by' + cz')\,x + (bx' + b'y' + c'z')\,y + (cx' + c'y' + c''z')\,z = 1.$$

Put OI $= r$, and let the tangent plane intersect OX, OY, OZ, in the points P, Q, R; then from this equation we have

$$\frac{1}{OP} = ax' + by' + cz' = r\,(a \cos.\alpha + b \cos.\beta + c \cos.\gamma) = rX,$$

$$\frac{1}{OQ} = bx' + b'y' + c'z' = r\,(b \cos.\alpha + b' \cos.\beta + c' \cos.\gamma) = rY,$$

$$\frac{1}{OR} = cx' + c'y' + c''z' = r\,(c \cos.\alpha + c' \cos.\beta + c'' \cos.\gamma) = rZ.$$

Now if p be the length of the perpendicular let fall from O on the tangent plane, the cosines of the angles which this perpendicular makes with the axes of coordinates will be equal to

$$\frac{p}{OP},\ \frac{p}{OQ},\ \frac{p}{OR},$$

respectively, that is, to prX, prY, prZ; and since the sum of the squares of these cosines is equal to unity, we have

$$pr\sqrt{X^2 + Y^2 + Z^2} = 1.$$

Hence it appears that the perpendicular let fall from O on the tangent plane is parallel to the direction of the resultant elastic force, and that the magnitude of the resultant is expressed by $\frac{1}{pr}$.

From this conclusion we may, with the greatest facility, deduce several corollaries.

1°. Since the elastic force is supposed finite, whatever be the *direction* of the displacement, it is manifest that the above-mentioned surface of the second order must be an ellipsoid, and that when the displacement is in the direction of any of the three axes of the ellipsoid, the elastic force excited will be in the direction of the same axis, because the tangent plane will be perpendicular to it. Hence the remarkable consequence, that *there are always three axes of elasticity at right angles to each other.*—(Memoir, p. 93.) Also the elasticities arising from equal displacements in the directions of the three axes are inversely as the squares of the axes; and hence, the positions of the axes, and the elasticities in their respective directions being given, the ellipsoid may be constructed.

2°. The ellipsoid being thus constructed, the direction of the elastic force, arising from a displacement in the direction of any of its semidiameters, will be parallel to the normal at the extremity of that semidiameter; and for equal displacements the magnitude of the force will be inversely as the rectangle under the semidiameter and the perpendicular from the centre on the tangent plane at its extremity. If the displacements are proportional to the semidiameters, the elastic forces will be both parallel and proportional to the normals; for the normal, terminated by any of the principal planes, is inversely as the perpendicular on the tangent plane.

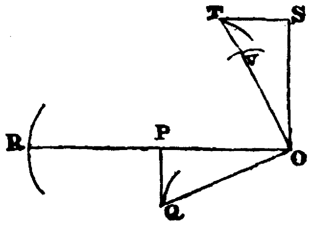
3°. If the elastic force be resolved into two, one parallel and the other perpendicular to the direction of the displacement, the former will be inversely as the square of the semidiameter in the direction of the displacement.

4°. If the ellipsoid be cut by a plane through its centre, and if the elastic force arising from a displacement in the direction of either axis of the section be resolved parallel and perpendicular to that axis, the part perpendicular to the axis will also be perpendicular to the plane of the section. For (by lem. 4.) the plane passing through one axis and the perpendicular to the tangent plane at its extremity, is perpendicular to the other axis, and therefore to the plane of the section. But if the displacement be in the direction of any other diameter of the section, the elastic force, resolved perpendicularly to that diameter, will be oblique to the plane of the section.

To apply these things to the double refraction of light in a crystallized medium, imagine the ellipsoid to be described as above, and let it be cut through its centre by a plane parallel to that of a *plane* wave of light incident on the crystal; then if the vibrations of the light be parallel to either of the axes of the section, the plane containing the direction of the vibrations and that of the elastic force arising from them will be perpendicular to the plane of the wave (No. 4, preceding); and therefore, according to Fresnel's theory, the direction of the vibrations will remain parallel to itself, while the wave is propagated. But if the light be common light, or if it be polarised, and the plane of polarisation be not perpendicular to either of the axes of the section, the wave will be divided into two others having the directions of their vibrations parallel to the semiaxes of the elliptic section, and their planes of polarisation perpendicular to them: their velocities of propagation—measured in a direction perpendicular to their plane—will be different, and will be in the subduplicate ratio of the elasticities in the direction of their respective vibrations, and therefore (by No. 3.) inversely as the semiaxes of the section to which those vibrations are parallel.

If a wave be propagated *in all directions* from an origin O within a crystal, its surface will at each instant be touched by the simultaneous position of a plane wave which passed through O at the instant when the former began to be propagated (Memoir, p. 127.) Hence, to find the surface of the double wave in a crystal, let the abovementioned ellipsoid be cut by any plane through its centre O, and imagine two other planes parallel to this section, and at distances from it which are third proportionals to its semiaxes OR, Or, and any given line k: the double surface which touches these planes in all their positions will be the surface of the wave.

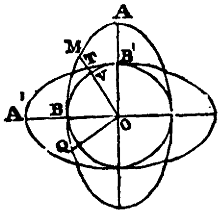
Now conceive another concentric ellipsoid, having the directions of its semiaxes the same, but their lengths a, b, c, inversely proportional to those of the former, so that the rectangle under any coinciding pair of semiaxes, is equal to k^2: then if a plane, touching this second ellipsoid in Q, cut OR perdendicularly in P, the line OP will be a third proportional to OR and k, (lem. 2.) and if Or intersect the second ellipsoid in q, the semiaxes of the section QOq will be OQ and Oq (lem. 4.) Draw OT perpendicular to QOq and equal to OQ, and conceive the surface which is the locus of the point T to be described, and a tangent plane, to which the line OS is drawn perpendicular, to be applied at the point T. Then OS will be perpendicular to the plane ROr and equal to OP (lem. 5.): and hence the point T always lies in the surface of the wave. Similar things may be proved with respect to the other semiaxis Or of the ellipse ROr. Hence we deduce the following construction for the surface of the wave:—



" Describe an ellipsoid whose axes are in the directions of the axes of elasticity, the squares of their lengths being *directly* as the elasticities in their respective directions; cut the ellipsoid by a plane through its centre, as QOq, and in a perpendicular to that plane take two portions OT and OV equal to the semiaxes OQ and Oq of the section. The double surface which is the locus of the points T and V is the surface of the wave."

As to the planes of polarisation of the rays belonging to the two parts of the wave, Fresnel has shewn how to find them by means of a surface which he calls the surface of elasticity. But it is desirable to be able to find them by means of the same ellipsoid which serves to find the surface of the wave. Now TS, parallel to OR, is the direction of the vibrations of the ray OT, and the tangent plane at Q is perpendicular to OR, and therefore parallel to the plane of polarisation of the ray OT. In like manner the plane of polarisation of the ray OV is parallel to the tangent plane at q. Hence the planes of polarisation of two different rays, having a common direction, are parallel to the planes which touch the ellipsoid at the extremities of the semiaxes of the diametral section perpendicular to their common direction.

When two of the axes are equal, the ellipsoid becomes a spheroid, and the crystal is said to be uniaxal, the double refraction being regulated by the third axis which is perpendicular to their plane. Let AOB be a section of the spheroid through the third axis OA, which is its axis of revolution; take OB′ = OB, and OA′ = OA, and let the ellipse A′OB′ and the circle BOB′ revolve about OB′ as an axis; they will describe a surface compounded of a spheroid and sphere, which will in this case be the surface of the double wave. For if OM be the direction of a ray, and if a plane perpendicular to OM cut the ellipse AOB in OQ, and the equator of the spheroid in Oq, the lines OQ and Oq, of which the latter is equal to OB, will be the semiaxes of the section QOq. Taking therefore OT and OV equal to OQ and OB, the locus of V will evidently be the circle BOB′; and the locus of T will be the ellipse A′OB′, since the angle B′OT is equal to the angle BOQ.



In the general case, OT and OV, which are equal to OQ and Oq, represent the velocities of the two different sorts of rays having a common direction in the crystal; and two right lines, which lie in the plane of the greatest and least axes of the ellipsoid, and are perpendicular to its two circular sections, are called the optic axes. It appears, therefore, by the 6th lemma, that the difference of the squares of the reciprocals of the velocities of the two rays having a common direction in the crystal, is proportional to the product of the sines of the angles which that direction makes with the optic axes. This is the celebrated law of M. Biot, to which he was led by analogy, and which he afterwards found to agree with that previously laid down by Dr. Brewster.

Such, in their simplest form, are the principal features of the Mechanical Theory of double Refraction, invented by the late M. Fresnel—a theory which would do honour to the sagacity of Newton, and which gives us ample reason to regret that the life of its author was not longer spared, to enrich with further discoveries his favourite science. Of him it may be said, as Newton said of Cotes—and apparently with much greater reason—that if he had lived longer, we should have known something at last of the laws of nature.

TRINITY COLLEGE, DUBLIN,
October 27, 1829.